SS 2016

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Differential Geometry

Homework 4

Mandatory Exercise 1. (10 points)
Recall that an action of a group G on a manifold M is called **proper** if the action map $G \times M \to M \times M$ is proper, i.e. the preimage of any compact set is compact.

(a) Show that if G is compact then the action is proper.

(b) How about if M is compact and G is not? Can you have a proper action of $\mathbb{R}$ on $S^1 \times S^1$?

Mandatory Exercise 2. (10 points)

(a) Give examples of matrices $A, B \in \mathfrak{gl}(2)$ such that $e^{A+B} \neq e^A e^B$.

For $A \in \mathfrak{gl}(n)$, consider the differentiable map

$$\begin{aligned} h\colon \mathbb{R} &\longrightarrow \mathbb{R} \setminus \{0\} \\ t &\longmapsto \det e^{At}. \end{aligned}$$

(b) This map is a group homomorphism between $(\mathbb{R}, +)$ and $(\mathbb{R} \setminus \{0\}, \cdot)$.

(c) Show that $h'(0) = \operatorname{tr} A$ and $\det(e^A) = e^{\operatorname{tr} A}$.

Suggested Exercise 1. (0 points)
Recall that the Lie algebra $\mathfrak{u}(2)$ of $U(2)$ consists of 2×2 skew-Hermitian matrices. For simplicity, we use multiplication by i to identify $\mathfrak{u}(2)$ with the set of 2×2 Hermitian matrices. The adjoint action of $U(2)$ is by conjugation:

$$A \cdot \xi = A \xi A^{-1},\ A \in U(n),\ \xi \in \mathfrak{u}(2).$$

Fix any real numbers $\lambda_1, \lambda_2 \in \mathbb{R}$, and let Λ denote the diagonal matrix $\operatorname{diag}(\lambda_1, \lambda_2)$.

(a) What is the orbit $U(2) \cdot \Lambda$ of Λ in $\mathfrak{u}(2)$?

(b) Does it depend on the choice of $\lambda_1, \lambda_2 \in \mathbb{R}$?

(c) Now consider the same question for the adjoint action of $U(n)$.

Suggested Exercise 2. (0 points)

(a) Show that $(\mathbb{R}, +)$ is a Lie group, determine its Lie algebra and write an expression for the exponential map.

(b) Prove that, if G is an abelian Lie group, then $[V, W] = 0$ for all $V, W \in \mathfrak{g}$.

(c) Find the Lie algebra of S^1 and compare it with the Lie algebra of $(\mathbb{R}, +)$. Can you deduce existence of some special map between $\mathbb{R}$ and S^1? Now do the same with $T^n = S^1 \times \ldots \times S^1$ (product of n circles) and $(\mathbb{R}^n, +)$.

Suggested Exercise 3. (0 points)
Consider the special linear group SL(2) consisting of 2×2 matrices with determinant 1. Show that SL(2) is a 3-manifold diffeomorphic to $S^1 \times \mathbb{R}^2$.

Suggested Exercise 4. (0 points)
Let $M = S^2 \times S^2$ and consider the diagonal S^1-action on M given by

$$e^{i\theta} \cdot (u, v) := \left(e^{i\theta} \cdot u, e^{2i\theta} \cdot v\right),$$

where, for $u \in S^2 \subset \mathbb{R}^3$ and $e^{i\beta} \in S^1$, $e^{i\beta} \cdot u$ denotes the rotation of u by an angle β around the z-axis.

(a) Determine the fixed points for this action.

(b) What are the possible non-trivial stabilizers?

Suggested Exercise 5. (0 points)
Let G be a Lie group and H a closed Lie subgroup, i.e. a subgroup of G which is also a closed submanifold of G. Show that the action of H in G defined by $A(h, g) = h \cdot g$ is free and proper.

Hand in: Monday 9th May
in the exercise session
in Seminar room 2, MI